THE STABILITY OF A SOLUTION OF A SYSTEM OF DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS RIGHT-HAND SIDES

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The stability of the periodic solution of the system of equations

$$\frac{dz}{dt} = f(z, t) \qquad (0.1)$$

with discontinuous periodic $[f(z, t + \tau) \equiv f(z, t)]$ right-hand sides [z and f are n-dimensional vector columns with coordinates z_i and $f_i (i = 1, \ldots, n)$] has been investigated by Aizerman and Gantmakher [1]. Establishing what should be understood by the linear approximation in this "discontinuous" case, the authors have proved theorems analogous to those of Liapunov.

The present paper deals with the stability of any solution (periodic or nonperiodic) of system (0.1) with discontinuous nonperiodic right-hand sides. For this, use is made of the condition for the discontinuities of the solution of the linear approximation introduced in paper [1] for periodic systems. Two criteria of stability are established which are generalizations of the corresponding theorems of Persidskii [2] and Perron [3], proved by these authors for continuous systems.

1. Conditions imposed on the right-hand sides of the differential equations. Consider the system of differential equations

$$\frac{dz}{dt} = f(z, t) \qquad (1.1)$$

where the real vector function $f(z, t)$ is given in the $(n + 1)$-dimensional space z, t inside a curvilinear cylinder C, the axis of which is the integral curve $z = z^0(t)$ of system (1.1). Let the infinite sequence of surfaces* $F_\alpha(z, t) = 0$ dissect the cylinder C into regions H_α, intersecting the curve $z = z^0(t)$ for $t = t_\alpha$ at points M_α. Then there exists a positive constant T such that $t_{\alpha+1} - t_\alpha \geqslant T > 0$.

* Here and in what follows the index α assumes the values $1, 2, \ldots, \infty$.

The planes $t = t_\alpha$ dissect the regions H_α into angular regions, bounded by these planes and the corresponding surfaces $F_\alpha = 0$, and into central regions, containing the segments of the integral curve $z = z^0(t)$. Concerning the function f and the surfaces $F_\alpha = 0$ the following assumptions are made:

1. The function f is continuous in every region H_α (including the boundaries $F_\alpha = 0$ and $F_{\alpha+1} = 0$), while passing through the surface $F_\alpha = 0$ it can experience only discontinuities of the first kind, the magnitudes ξ_α of which at points M_α are bounded in their totality.

2. Conditions are fulfilled which guarantee in every region H_α uniqueness of the solution of system (1.1) for the given initial conditions and its continuous dependence on these conditions. Also satisfied are the conditions for the continuation of the integral curves without any obstacles from any region H_α into the adjacent region $H_{\alpha+1}$.

3. In every central region

$$f(z, t) = f[z^\circ(t), t] + P(t)[z - z^\circ(t)] + R(z, t) \tag{1.2}$$

holds, where $P(t)$ is continuous in every interval* $t_\alpha \leqslant t \leqslant t_{\alpha+1}$, while the matrix $R(z, t)$, which is bounded for $t \geqslant 0$ and represents the nonlinear remainder, satisfying the inequality

$$|R(z, t)| < a\,|z - z^\circ(t)| \qquad (t \geqslant 0,\ a = \text{const}) \tag{1.3}$$

Here and in what follows $|z| = (z_1^2 + \ldots + z_n^2)^{1/2}$.

4. The limit relation

$$f(z, t) - f[z^\circ(t), t] \to \xi_\alpha, \quad \text{и}\ f(z, t) - f[z^\circ(t), t] \to -\xi_\alpha \quad \text{for } (z, t) \to M_\alpha$$

which holds in any angular region below and above the plane $t = t_\alpha$, is fulfilled uniformly with respect to α.

5. The surfaces $F_\alpha = 0$ are continuous and at points M_α they are smooth. On one side of the surface $F_\alpha = 0$ we have $F_\alpha > 0$ while on the other side $F_\alpha < 0$ holds. Inside the cylinder C the surfaces $F_\alpha = 0$ do not intersect each other.

6. Along the integral curve $z = z^0(t)$ we have

$$\left(\frac{dF_\alpha}{dt}\right)^- \neq 0, \qquad \frac{(dF_\alpha / dt)_{M_\alpha^+}}{(dF_\alpha / dt)_{M_\alpha^-}} \geqslant \Gamma > 0 \quad (\Gamma = \text{const}) \tag{1.4}$$

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* Speaking about intervals $t_\alpha \leqslant t \leqslant t_{\alpha+1}$, we shall have in mind also the interval $0 \leqslant t \leqslant t_1$

Here

$$\frac{dF_\alpha}{dt} = \left(\frac{dF_\alpha}{\partial z} f + \frac{\partial F_\alpha}{\partial t}\right)_{z=z^0(t)}$$

where $dF_\alpha/\partial z$ denotes the vector gradient (row) and the indices + and − refer correspondingly to the values for $t = t_\alpha + 0$ and $t = t_\alpha - 0$.

According to (1.4) the equations for the parts of the surface $\Phi_\alpha(x, t) \equiv F_\alpha(z^0 + x, t) = 0$, situated below and above the plane $t = t_\alpha$ can be written correspondingly in the form $(x = z - z^0(t))$

$$t_\alpha - t = h_\alpha^- x + 0(|x|), \qquad t - t_\alpha = h_\alpha^+ x + 0(|x|) \tag{1.5}$$

where the vector row is given by

$$h_\alpha^\pm = \left[\frac{\partial F_\alpha}{\partial z} \Big/ \left(\frac{dF_\alpha}{dt}\right)^\pm\right]_{M_\alpha}$$

7. The quantities h_α^- are bounded in their totality. The ratio in (1.5)

$$\frac{0(|x|)}{|x|} \to 0 \quad \text{for } |x| \to 0 \tag{1.6}$$

is satisfied uniformly with respect to α.

2. Linear approximation and its transformation. Let us define the linear approximation of system (1.1) as the set of : (i) the system of linear equations

$$\frac{dx}{dt} = \mathrm{P}(t)\, x \tag{2.1}$$

which is satisfied by the solution $x = x(t)$ inside every interval $t_\alpha \leqslant t \leqslant t_{\alpha+1}$, and (ii) the conditions of discontinuities at $t = t_\alpha$ of the integral curves $x = x(t)$, defined by the formulas

$$x_\alpha^+ = S_\alpha x_\alpha^- \tag{2.2}$$

where the matrix

$$\mathrm{S}_\alpha = \|(\mathrm{S}_\alpha)_{ik}\|_1^n, \qquad (\mathrm{S}_\alpha)_{ik} = \delta_{ik} + \xi_{\alpha i} h_{\alpha k}^-$$

δ_{ik} is the Kronecker symbol, $h_{\alpha k}^-$ and $\xi_{\alpha i}$ are the corresponding coordinates of the vectors h_α^- and ξ_α. The matrices S_α are bounded in their totality.

For the proof of the criterion of stability according to the linear approximation the following lemma will be needed.

Lemma. For every system of linear approximation (2.1) + (2.2) it is possible to construct a Liapunov transformation discontinuous at

$t = t_\alpha$*

$$x = L(t)\,y \tag{2.3}$$

which transforms this system into a system the matrix of whose coefficients $A(t)$ is continuous and bounded for $t \geqslant 0$ and the solutions of which are continuous:

$$\frac{dy}{dt} = A(t)\,y, \qquad y_\alpha^+ = y_\alpha^- \tag{2.4}$$

Proof. Let the values of $L(t)$ and dL/dt for $t = t_\alpha \pm 0$ be given by the formulas

$$L_\alpha^- = E, \qquad (dL/dt)_\alpha^- = 0 \qquad (E = \|\delta_{ik}\|_1^n) \tag{2.5}$$

$$L_\alpha^+ = S_\alpha \qquad (dL/dt)_\alpha^+ = P_\alpha^+ S_\alpha - S_\alpha P_\alpha^- \tag{2.6}$$

The relations (2.6) guarantee the continuity of the matrix $A(t)$ and of the solutions $y = y(t)$ of the system (2.4) for $t = t_\alpha$.

For the proof of the Lemma it is sufficient to construct a matrix $L(t)$ according to the given values (2.5) and (2.6) for $t = t_\alpha \pm 0$ of this matrix and its derivative in such a way that in every interval $t_\alpha \leqslant t \leqslant t_{\alpha+1}$ there exist continuous matrices $L^{-1}(t)$ and dL/dt, bounded for $t > 0$ in the same way as $L(t)$.

For the existence and boundedness for $t > 0$ of the matrix $L^{-1}(t)$ it is sufficient that the matrix $L(t)$ satisfies the relation

$$\det L(t) \geqslant \Gamma > 0 \qquad (t \geqslant 0) \tag{2.7}$$

This condition is satisfied for $t = t_\alpha + 0$ by virtue of (1.4), since from the structure of the matrix S_α follows (see [1], p. 662) that

$$\det L_\alpha^+ = \det S_\alpha = \frac{(dF_\alpha/dt)_{M_\alpha^+}}{(dF_\alpha/dt)_{M_\alpha^-}}$$

holds.

The condition (2.7) is also satisfied for $t = t_\alpha - 0$, if we assume that in (1.4) we have $\Gamma < 1$.

Let us pass now to the determination of the values of the matrix $L(t)$ inside the intervals $t_\alpha \leqslant t \leqslant t_{\alpha+1}$. For this, consider the column s_k of the matrix S_α as a vector in an n-dimensional space and take a parallelepiped constructed at the origin of the coordinates on the vectors

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* Except for the discontinuities at $t = t_\alpha$ the properties of the matrix $L(t)$ are the same as in the classical case, i.e. in every interval $t_\alpha \leqslant t \leqslant t_{\alpha+1}$ there exist continuous matrices L^{-1} and dL/dt, which are bounded for $t \geqslant 0$ in the same way as L.

$s_1, \ldots, s_n$ (as edges). Change continuously the coordinates of these vectors, keeping their lengths constant, and increasing at the same time the volume of the parallelepiped in such a way that for $t = t_{\alpha 1} = t_\alpha + 1/4(t_{\alpha+1} - t_\alpha)$ the parallelepiped becomes rectangular.

Take the current values of the coordinates l_{ik} of the vectors s_k in this transformation for the elements of the matrix $L(t) = \| l_{ik}(t) \|_1{}^n$ in the corresponding intervals $t_\alpha \leqslant t \leqslant t_{\alpha 1}$. Then for $t_\alpha \leqslant t \leqslant t_{\alpha 1}$ we have

$$\sum_{i=1}^{n} | l_{ik} |^2 \leqslant r^2, \quad \det \mathrm{L}(t) > \Gamma > 0, \quad (i, k = 1, \ldots, n;\ 1 < r < \infty,\ \mathrm{r = const}) \quad (2.8)$$

Keep for the current values of the coordinates of the vectors the previous notations $l_{ik}(t)$ and take them for the elements of the matrix $L(t)$. In addition carry out the following three transformations:

(1) In the intervals of time $t_{\alpha 1} \leqslant t \leqslant t_{\alpha 2} = t_\alpha + 1/2(t_{\alpha+1} - t_\alpha)$, by stretching the edges to the length r, convert the rectangular parallelepipeds into cubes;

(2) In the time intervals $t_{\alpha 2} \leqslant t \leqslant t_{\alpha 3} = t_\alpha + 3/4(\ _{\alpha+1} - t_\alpha)$ turn the cubes so that their edges become parallel to the coordinate axes;

(3) And, finally, during the time intervals $t_{\alpha 3} \leqslant t \leqslant t_{\alpha+1}$, compress the lengths of the edges to unit lengths. In this way the condition $L^{-1} = E$ is satisfied.

In the above transformations we connected, by continuous arcs of curves and segments of straight lines, pairs of points of the n-dimensional space in such a way that the inequalities (2.8) always hold.

From the actual process of construction of these arcs and segments it follows that they are of bounded lengths for all α. Since, in addition, the time during which these arcs are described is greater than (1/4) $T > 0$, then the description of these arcs can be carried out with velocities the magnitudes of which are bounded by one and the same constant number for all α. The motion along the arcs can be started and ended in every interval with zero velocities.

In order that the matrix dL/dt assumes for $t = t_\alpha + 0$ the values given by the formulas (2.6), replace the graphs of these functions $l_{ik} = l_{ik}(t)$ $(i, k = 1, \ldots, n)$ by the nearest smooth curves which coincide with the initial curves for $t = t_\alpha + 0$ and in the intervals $t_\alpha + t/2 \leqslant t \leqslant t_{\alpha+1}$ in such a way that the functions for which we kept the previous notations $l_{ik}(t)$, but which represent the new curves, satisfy the equalities (2.6). Since the magnitude of the determinant det $L(t)$ is a continuous function of its elements, and the matrices $(dL/dt)_\alpha{}^+$ are bounded in their totality, then the new curves can be traced in such a way that the inequalities

$$|l_{ik}(t)| \leqslant 2r, \qquad \det \mathrm{L}(t) > {}^1/_2\Gamma > 0 \qquad (i, k = 1, \ldots, n;\ t \geqslant 0)$$

are fulfilled and that at the same time the boundedness of the matrix dL/dt for $t \geqslant 0$ is not violated. Hence the Lemma is proved.

Remark. Usually for the construction of the Liapunov transformation the knowledge of the solutions of the corresponding system of differential equations is necessary (see [4-6]). In the case under consideration, however, for the construction of the transformation $x = L(t)y$, it is sufficient to give only the matrices S_α and $P(t_\alpha \pm 0)$.

3. Criteria of stability according to the linear approximation.

Theorem 1. Let the elements of the normalized (for $t = t_0$) fundamental matrix $\|x_{ik}(t, t_0)\|_1^n (x_{ik}(t, t_0) = \delta_{ik})$ of the system of linear approximation (2.1) + (2.2) satisfy for arbitrary $t_0 \geqslant 0$ and $t \geqslant t_0$ the relations

$$|x_{ik}(t, t_0)| < B \exp[-\beta(t - t_0)] \qquad (i, k = 1, \ldots, n) \tag{3.1}$$

where B and β are positive constants which do not depend on t_0. Then the solution $z = z^0(t)$ of the initial nonlinear system (1.1) is asymptotically stable provided only that the constant a in the inequality (1.3) is sufficiently small.

In order to prove the Theorem apply to the initial nonlinear system (1.1), rewritten in terms of the variations $x = z - z^0(t)$, the transformation (2.3). Then we obtain the system

$$\frac{dy}{dt} = q(y, t), \qquad q(y, t) = \mathrm{L}^{-1}\left[f(z^\circ + \mathrm{L}y, t) - f(z^\circ, t) - \frac{dL}{dt}y\right] \tag{3.2}$$

which has discontinuous solutions for $t = t_\alpha$, since for these values the matrix $L(t)$ is discontinuous. In the space y, t the surfaces $Q_\alpha(y, t) \equiv \Phi_\alpha(Ly, t) = 0$ and the planes $t = t_\alpha$ dissect the cylinder C into angular and central regions in the same way as in the space z, t.

In terms of the variables y the system of linear approximation (2.1) + (2.2) can be rewritten in the form (2.4). From the relations (3.1) and the boundedness of the matrices L and L^{-1} follows that the elements of the normalized (at $t = t_0$) fundamental matrix $\|y_{ik}(t, t_0)\|_1^n$ of system (2.4) satisfy for arbitrary $t_0 \geqslant 0$ and $t \geqslant t_0$ the inequalities

$$|y_{ik}(t, t_0)| < B_1 \exp[-\beta(t - t_0)] \qquad (i, k = 1, \ldots, n)$$

where B_1 and β are positive constants which do not depend on t_0. If these last relations are satisfied, then, as it was proved by Malkin [5], there exist a positive definite quadratic form $V(y, t)$ with continuous and bounded coefficients which satisfies the relations

$$\left(\frac{\partial V}{\partial y}\right) Ay + \frac{\partial V}{\partial t} = -|y|^2 \tag{3.3}$$

$$b_1 |y|^2 \leqslant V(y, t) \leqslant b_2 |y|^2 \qquad (t \geqslant 0,\ 0 < b_1 < 1 < b_2) \tag{3.4}$$

Let us investigate the change of values of $V(y, t)$ along the discontinuous integral curves $y = y(t)$ of the system (3.2).

1. In a central region. By virtue of (1.2), (1.3) and (3.2) we have

$$g(y, t) = A(t)\, y + R^*(y, t), \qquad R^*(y, t) = L^{-1} R(Ly + z^\circ, t), \qquad |R^*| < a_1 |y|$$

where $a_1 = a\varkappa$, $\varkappa > 0$ and finite due to the boundedness of L and L^{-1}. Denoting by V' the total derivative of V with respect to t, evaluated by means of the equations (3.2) and the relations (3.3) and (3.4), we obtain

$$\frac{V'}{V} = -\frac{1}{V}|y|^2 + \frac{1}{V}\frac{\partial V}{\partial y} R^* \leqslant -\frac{1}{b_2} + \frac{a_2}{b_1}$$

where $a_2 = a_1 \sup|\partial V/\partial y|\, |y|^{-1}$ for $t \geqslant 0$. Since the coefficients of the form V are bounded, then a_2 is a finite quantity. Supposing the constant a in (1.3) to be so small that $a_2 b_2 < b_1$, we obtain

$$\frac{V'}{V} \leqslant -\mu^2 \qquad \left(\mu^2 = \frac{1}{b_2} - \frac{a_2}{b_1}\right) \tag{3.5}$$

From (3.5) follows that the values of V at instants t and t^* ($t^* < t$), when the point of the integral curve is in one of the central regions, satisfy the relation

$$V \leqslant V^* \exp[-\mu^2 (t - t^*)] \tag{3.6}$$

2. In an angular region. Applying the estimating scheme, analogous to that used in paper [1], and taking into account the properties of the form V and the conditions 4, 6 and 7 (Section 1), we obtain that the values of V at instants t and t^{**}, when the point of the integral curve is in one and the same angular region, satisfy for sufficiently small y the inequality

$$V < NV^{**} \quad (N > 1) \tag{3.7}$$

where N does not depend on a.

Moreover, if the integral curve passes from the point y_1, t_1 on the surface of discontinuity $Q_\alpha = 0$ to the point y_α, t_α on the plane $t = t_\alpha$, then for sufficiently small y the double inequality

$$\exp(-\theta) < \frac{V(y_\alpha{}^+, t_\alpha)}{V(y_1, t_1)} < \exp\theta \tag{3.8}$$

holds, where θ is an arbitrarily small positive number.

Let it be given that $\epsilon > 0$ and so small that for $|y|^2 \leqslant \epsilon$ the inequalities (3.6), (3.7), (3.8) are satisfied and the time Δt spent in any angular region (inside the cylinder $|y|^2 = \epsilon$) is less than $(1/2\,\mu^2)T(\mu^2 - \nu^2)$, where $0 < \nu < \mu$. Then inside the cylinder $|y|^2 = \epsilon$ the planes $t = t_\alpha{}^* = 1/2(t_{\alpha+1} + t_\alpha)$ do not intersect angular regions.

Select $\delta = \epsilon b_1/N$, $\theta < 1/2(\mu^2 - \nu^2)T$ and the initial point $(y_0, t_1{}^*)$ of the integral curve in such a way that $b_2|y_0|^2 < \delta$. Then by virtue of (3.4) we have

$$V_1 = V_{t=t_1^*} \leqslant b_2\,|\,y_0\,|^2 < \delta = \frac{\varepsilon b_1}{N}$$

From the inequalities (3.6) and (3.7) follows that in the interval $t_1{}^* \leqslant t \leqslant t_2{}^*$ the rate of increase of the function $V(y, t)$ does not exceed N. Therefore, in the whole interval we have $V(y, t) < \epsilon b_1$, and in confirmity with (3.4) the inequality $|y|^2 < \epsilon$ holds.

Using the inequalities (3.6) and (3.8) and the relations $t_{\alpha+1} - t_\alpha \geqslant T$, we obtain

$$V_2 = V_{t=t_2^*} \leqslant V_1 \exp\left[-\mu^2(T - \Delta t) + \theta\right] < V_1 \exp\left(-\nu^2 T\right)$$

i.e. $V_2 < V_1 < \delta$. Therefore, the arguments used above can be repeated for the interval $t_2{}^* \leqslant t \leqslant t_0{}^*$, and so on.

Consequently, any integral curve of the nonlinear system (3.2), which has started for $t = t^*$ inside the cylinder $|y|^2 = \delta/b_2$, will remain all the time inside the cylinder $|y|^2 = \epsilon$ and

$$V_\alpha = V_{t=t_\alpha^*} \leqslant V_1 \exp\left[-(\alpha - 1)\nu^2 T\right]$$

Therefore, in every interval $t_\alpha{}^* \leqslant t \leqslant t^*_{\alpha+1}$ we have

$$b_1\,|\,y\,|^2 \leqslant V \leqslant NV_\alpha \leqslant Nb_2\,|\,y_0\,|^2 \exp\left[-(\alpha - 1)\nu^2 T\right.$$

and $y \to 0$ for $t \to \infty$. Hence the theorem is proved.

Theorem 2. Let $\omega(t)$ be an arbitrary vector function, bounded for $t \geqslant 0$ and piecewise continuous and with discontinuities only at $t = t_\alpha$: $\omega_\alpha{}^+ = S_\alpha \omega_\alpha{}^-$. Further let any solution of the system

$$\frac{dx}{dt} = \mathrm{P}(t)\,x + \omega(t) \tag{3.9}$$

satisfying these equations inside every interval $t_\alpha \leqslant t \leqslant t_{\alpha+1}$ and experiencing discontinuities at $t = t_\alpha$ for which $x_\alpha{}^+ = S_\alpha x_\alpha{}^-$ be bounded for $t \geqslant 0$. Then the solution $z = z^0(t)$ of the nonlinear system (1.1) is asymptotically stable provided only that the constant a in inequality (1.3) is sufficiently small.

In order to prove the theorem, apply to the system (3.9) the transformation (2.3) and afterwards to the so obtained system the transformation of Perron [6] $y = L_1 u$ (L_1 is a continuous Liapunov matrix). Then the system is finally reduced to the form

$$\frac{du}{dt} = G(t)\,u + \omega^*(t) \qquad (\omega^* = L_1^{-1}L^{-1}\omega) \tag{3.10}$$

where $G(t) = \|g_{ik}\|_1^n$ is a triangular matrix bounded and continuous for $t \geqslant 0$, i.e. $g_{ik}(t) \equiv 0$ for $k > i$.

From the boundedness of all the solutions of (3.9) follows the boundedness of any solution of (3.10) for arbitrary continuous bounded $\omega^*(t)$ ($t > 0$). According to Perron [3] this implies the boundedness of all $2n$ functions

$$\exp\int_{t_0}^{t} g_{ii}(\tau)\,d\tau, \qquad \exp\left[\int_{t_0}^{t} g_{ii}(\tau)\,d\tau\right]\int_{t_0}^{t}\exp\left[-\int_{t_0}^{\tau} g_{ii}(t)\,dt\right]d\tau \quad (i = 1,\ldots, n) \tag{3.11}$$

If, however, the functions (3.11) are bounded, then, as it was proved by Malkin [7], there exists a positive definite function, admitting an infinitely small upper limit, the total time derivative of which by virtue of the system

$$\frac{du}{dt} = G(t)\,u \tag{3.12}$$

is a negative definite function.

Then, according to a theorem of Fersidskii [2], the elements of the normalized (at $t = t_0$) fundamental matrix $U(t, t_0) = \|u_{ik}(t, t_0)\|_1^n$ of the system (3.12) satisfy for arbitrary $t_0 \geqslant 0$, $t \geqslant t_0$ the relations

$$|u_{ik}(t, t_0)| < B_1 \exp[-\beta(t - t_0)] \qquad (i, k = 1, \ldots, n) \tag{3.13}$$

where B_1 and β are positive constants which do not depend on t_0.

If $X(t, t_0) = \|x_{ik}(t, t_0)\|_1^n$ is the normalized (at $t = t_0$) fundamental matrix of the system (3.9) for $\omega(t) \equiv 0$, then from (3.13) we obtain for arbitrary $t_0 > 0$, $t \geqslant t_0$ the inequalities

$$|x_{ik}(t, t_0)| \leqslant B \exp[-\beta(t - t_0)] \quad (i, k = 1, \ldots, n)$$

where B and β are positive constants which do not depend on t_0.

In this way, if the conditions of Theorem 2 are satisfied, also the conditions of Theorem 1 hold. This proves Theorem 2.

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